

The effect of service time variability on maximum queue lengths in $M^X/G/1$ queues

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Abstract

We study the impact of service-time distributions on the distribution of the maximum queue length during a busy period for the $M^X/G/1$ queue. The maximum queue length is an important random variable to understand when designing the buffer size for finite buffer ($M/G/1/n$) systems. We show the somewhat surprising result that for three variations of the preemptive LCFS discipline, the maximum queue length during a busy period is smaller when service times are more variable (in the convex sense).

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1 Introduction

An important design issue for telecommunication systems and other applications is determining the buffer size when buffers are finite. We can better understand the effect of a particular buffer size by understanding the distribution of the maximum queue length during a busy period in an infinite buffer system. We give a characterization of the busy-period maximum queue length, M , for the $M^X/G/1$ queue for three types of preemptive LCFS (last-come first-served) disciplines: (i) preempted services are resumed when service recommences (LCFS-p-resume), (ii) preempted services must be restarted from scratch when service recommences and a new service time is chosen from the service-time distribution (LCFS-p-repeat with resampling), and (iii) preempted services must be restarted from scratch when service recommences but the total service requirement for a given customer is the same each time it restarts its service (LCFS-p-repeat-without-resampling). These characterizations of M for each of the queueing disciplines allow us to show the effect of service-time distributions on M , as stated in (i)-(iii) below. For a fixed service discipline, let M and M' be the maximum number of customers during a busy period in two $M^X/G/1$ queues with respective generic service times S and S' , and with the same arrival rate λ , and the same batch-size distribution. We assume that the distributions of S and S' are such that the queues are stable. In this paper we show that the following relations hold; the definitions of the various stochastic orders can be found in the next section.

- (i) Under the LCFS-p-resume discipline, if $S' \leq_{LT} S$, then $M' \leq_{st} M$.
- (ii) Under the LCFS-p-repeat (with resampling) discipline, if $E(e^{-\lambda S'}) \geq E(e^{-\lambda S})$, then $M' \leq_{st} M$.
- (iii) Under the LCFS-p-repeat-without-resampling discipline, if $S' \leq_{icv} S$, then $M' \leq_{st} M$.

A consequence of our results is the somewhat surprising conclusion that M will be stochastically smaller when service times are more variable (in the convex sense) under the preemptive LCFS disciplines. Miyazawa (1990) and Miyazawa and Shanthikumar (1991) show that for the finite-buffer $M^X/G/1/n$ queue under a non-preemptive discipline, the loss rate, i.e., the probability that a random customer is lost, will be larger when service times are more variable in the convex sense. Our result relates to the loss rate, but the effect goes in the other direction. That is, we have that for preemptive LCFS disciplines, $P(M > n)$ is smaller when service times are larger in the convex sense, where $P(M > n)$ can be interpreted as the probability of at least one loss during a busy period in the $M^X/G/1/n$ queue. See also Chang, Chao, Pinedo, and Shanthikumar (1991).

For other results on the impact of the service time and batch size distributions on various performance measures of queueing systems, see, for example, Hordijk (2001), Makowski (1994), and Shanthikumar and Yao (1994), and the references therein. For other applications of the preempt-repeat service discipline, see, e.g., Adiri, Frostig, and Rinnooy Kan (1991), Birge, Frenk, Mittenthal, and Rinnooy Kan (1990), Cai, Sun, and Zhou (2004), and Cai, Wu, and Zhou (2004).

The paper is organized as follows. We first recall some definitions of stochastic ordering in the next section. We then study M for each of the preemptive LCFS disciplines. Finally we provide some numerical illustrations of our results.

2 Preliminaries

Recall the following stochastic ordering relations for random variables X and Y .

Definition 2.1 *X is larger than Y in the stochastic sense, $X \geq_{st} Y$, if $E\phi(X) \geq E\phi(Y)$ for all increasing functions ϕ for which the expectations exist.*

Equivalently, $X \geq_{st} Y$ if and only if $P(X > t) \geq P(Y > t)$ for all t .

Definition 2.2 *X is larger than Y in the convex sense, $X \geq_{cx} Y$, if $E\phi(X) \geq E\phi(Y)$ for all convex functions ϕ for which the expectations exist.*

Note that $X \geq_{cx} Y$ implies $EX = EY$ and $Var(X) \geq Var(Y)$. In this sense the convex ordering is an ordering of variability in random variables.

Definition 2.3 *X is larger than Y in the increasing concave sense, $X \geq_{icv} Y$, if $E\phi(X) \geq E\phi(Y)$ for all increasing concave functions ϕ for which the expectations exist.*

Definition 2.4 *X is larger than Y in the Laplace-transform sense, $X \geq_{LT} Y$, if $E[e^{-\theta X}] \leq E[e^{-\theta Y}]$ for all $\theta > 0$ for which the expectations exist.*

Note that $X \geq_{cx} Y$ implies $X \leq_{icv} Y$, which in turn implies $X \leq_{LT} Y$.

Finally, for reasons of brevity we use the following notation. When we say $X = [Y \mid Z = z]$, we mean that $P(X = x) = P(Y = x \mid Z = z)$ for all x .

3 Preemptive LCFS disciplines

3.1 LCFS preempt-resume

We first consider the $M^X/G/1$ queue with the LCFS preempt-resume (LCFS-p-resume) discipline. That is, the customer that has been in the system the least amount of time is always served, and newly arriving customers preempt earlier arrivals already in service. Within a batch customers are arbitrarily labeled, so that we may think of them as arriving sequentially, though immediately after each other. Thus, one customer in a newly arriving batch will be considered the most recent arrival and will immediately enter service, and the rest of the batch cannot be served until that customer, as well as all customers arriving in later batches that preempt that customer, are served.

Customers who resume service after being preempted start their service where they left off. Hence, a random service with service time S that is preempted when t units of service have already been received has remaining service time $[S - t | S > t]$. We also assume service is non-idling. Let T be a generic interarrival time, where T has an exponential distribution with rate λ , and let X be a generic batch size with arbitrary distribution and mean μ . We assume that the queue is stable, $\lambda\mu ES < 1$.

Let customer 0 be the last customer in the first batch in the busy period, i.e., the first customer to enter service, and let S_0 be the service time of customer 0. Let $N = N(S_0)$ be the number of Poisson batch arrival times that occur during the service of customer 0, and let $N(s) = [N(S_0) | S_0 = s]$. Note that the service will be interrupted if $N(S_0) > 0$. Let X_0 be the number of customers in the first batch of the busy period and define $M(k, n) = [M | X_0 = k, N = n]$ and $M(k) = [M | X_0 = k]$, so that $M(X_0, N) = M = M(X_0)$. Let $M_i, i = 1, 2, \dots$, be i.i.d. copies of M , and define $\max_{i=1, \dots, n} M_i$ to be 0 if $n = 0$. For the LCFS-p-resume discipline we then have the following characterization of $M(k, n)$.

Theorem 3.1 *The maximum queue length $M(k, n)$ for the $M^X/G/1$ queue under the LCFS-p-resume discipline satisfies*

$$M(k, n) =_d \max\left\{\max_{i=1, \dots, n} M_i + k; M(k-1)\right\}, \quad k \geq 1, n \geq 0, \quad (1)$$

where $M(0) = 0$, and $M_i, i = 1, 2, \dots$, and $M(k-1)$ are independent.

Proof. We can think of constructing the busy period, conditional on $X_0 = k, S_0 = s$ and $N(s) = n$, as follows. Denote the arrival epochs, on a clock that only ticks when customer 0 is being served, by $0 < t_1 < \dots < t_n < s$. A batch of customers arrives at time t_1 and starts a

new independent busy period (and stops our clock temporarily), except that there are k more customers in the queue (the original customers) throughout that busy period. When this first sub-busy period is over, at time $t_1 + \tau$ say, then customer 0 returns to service and our clock resumes ticking. Another batch arrives at time $t_2 + \tau$, starting a new independent busy period, and so on, until the n sub-busy periods have completed, as well as the original service time s . Then a new busy period starts with the other $k - 1$ customers that arrived in the first batch, and the maximum queue length during that busy period has the same distribution as $M(k - 1)$. Because the arrival process is memoryless, this construction is stochastically equivalent to the dynamics of a generic $M^X/G/1$ busy period starting with k customers. \square

Let $P(k, b) = P(M(k) \leq b)$, and $P(b) = P(M \leq b) = EP(X_0, b)$. So $P(0, b) = 1$ and $P(0) = 0$. Using the fact that $E[P(b - k)^N]$ is the z -transform, or probability generating function of N evaluated at $z = P(b - k)$, we have from Theorem 3.1 that for $1 \leq k \leq b$,

$$\begin{aligned} P(k, b) &= E[P(M + k \leq b)^N]P(k - 1, b) \\ &= E[P(b - k)^N]P(k - 1, b) \\ &= Ee^{-\lambda(1 - P(b - k))S}P(k - 1, b). \end{aligned}$$

Corollary 3.2 *For $b \geq k \geq 1$,*

$$P(k, b) = \prod_{i=1}^k Ee^{-\lambda(1 - P(b - i))S}.$$

If we restrict ourselves to unit batch sizes only, so $X \equiv 1$ and $P(b) = P(1, b)$, we have the following corollary.

Corollary 3.3 *If $X \equiv 1$, then for $b \geq 1$,*

$$P(b) = Ee^{-\lambda(1 - P(b - 1))S}.$$

Now we can see how the distribution of S affects M .

Theorem 3.4 *For $M^X/G/1$ queues operating under the LCFS- p -resume discipline, if $S' \leq_{LT} S$, then $M' \leq_{st} M$. In particular, if $S' \geq_{cx} S$, then $M' \leq_{st} M$.*

Proof. To show that $M' \leq_{st} M$, we show that $P(k, b) \leq P'(k, b)$ (with the obvious definition for P') for all k and b by induction on b and k . For each b we have $P(0, b) = 1 = P'(0, b)$, and $P(k, 1) = 0 = P'(k, 1)$ for $k > 1$. Since $S' \leq_{LT} S$,

$$P(1, 1) = P(S < T) = Ee^{-\lambda S} \leq Ee^{-\lambda S'} = P'(1, 1).$$

Suppose $P(i, a) \leq P'(i, a)$ for $a < b$ and all $i \geq 0$, so $P(a) \leq P'(a)$ for all $a < b$, and suppose $P(i, b) \leq P'(i, b)$ for all $0 \leq i < k$, and consider b and k . From Corollary 3.2, the induction hypothesis, and the assumption $S' \leq_{LT} S$, it then follows that

$$\begin{aligned} P(k, b) &= \prod_{i=1}^k Ee^{-\lambda(1-P(b-i))S} \leq \prod_{i=1}^k Ee^{-\lambda(1-P'(b-i))S} \\ &\leq \prod_{i=1}^k Ee^{-\lambda(1-P'(b-i))S'} = P'(k, b). \end{aligned}$$

This completes the proof. \square

This theorem is illustrated by Figures 1, 2 and 3 below, showing the probabilities $P(M \leq n)$ for (convexly ordered) families of uniform, Pareto and hyperexponential distributions.

Remark It is well known (Kelly, 1979) that the $M/G/1$ queue under the LCFS-p-resume discipline exhibits service time insensitivity in the sense that the marginal distribution of the number in the stationary system, L , depends on the service-time distribution only through its mean. At first this seems at odds with our results, but we must bear in mind that the maximum number in the system during a busy period depends on the sample-path evolution of the queue length over a busy period. Hence the behaviour of M and L may be very different. This idea is further illustrated by the following heuristic example.

Example Let M be the maximum number in system for an $M/G/1$ LCFS-p-resume queue with $S \equiv 1$ (call this system 1) and let M' be the corresponding maximum when the first service time in a busy period, S' , is equally likely to be ε or $2 - \varepsilon$ so $S \leq_{cx} S'$, and the other service times in the busy period are identically equal to 1 (call this system 2). Then, for ε very small, the first busy period in system 2 is equally likely to be very short and have a maximum of 1, or it will essentially consist of two busy periods, each evolving as a busy-period in system 1. The second of these busy periods starts when the initial customer has received $1 - \varepsilon/2$ service. That is, roughly, M' is equally likely to be 1 or to have the same distribution as $\max\{M_1, M_2\}$, so $M' \neq_{st} M$. Note however that L and L' have roughly the same distribution. Indeed, $P(L = 0) = P(L' = 0)$, since the workload is the same in both systems. Furthermore, a random arrival during a busy period in system 2 will either see a customer with $S' = \varepsilon$ in service, with very small probability, or will arrive during one of the two busy periods that each evolve as in system 1. Hence L' and L have roughly the same distribution. Finally note that the distribution of the length of a busy period does depend on the distribution of S . \square

The $M^X/G/1/b$ LCFS-p-resume queue also exhibits insensitivity, i.e., the distribution of the number in system, L_b , depends on the distribution of S only through its mean. Hence, the

loss rate in the $M^X/G/1/b$ queue, $P(L_b = b)$, is insensitive to the distribution of S . In contrast, our result shows that the probability of at least one loss during a busy period, $P(M > b)$, does depend on the distribution of S , and is greater when S is larger in the Laplace-transform sense.

3.2 LCFS preempt-repeat with resampling

Now we suppose that when services are preempted they must be restarted from scratch. The new service time is assumed to be an independent random variable with the same distribution. We call this the LCFS-p-repeat (with resampling) discipline. Of course, the behavior of the queue under the LCFS-p-resume and LCFS-p-repeat disciplines is the same when service times are exponential.

We use the same notation as in the previous subsection. Now, for stability, we need $\lambda\mu ES_e(S) < 1$ and $\lambda\mu ES_e(S') < 1$, where $S_e(S)$ is the effective service time, i.e., the total time a random customer must spend in service, including restarts due to interruptions. Thus,

$$ES_e(S) = E(S \wedge T) + P(S > T)ES_e(S),$$

where $a \wedge b = \min\{a, b\}$, and hence

$$ES_e(S) = \frac{E(S \wedge T)}{P(S \leq T)}. \quad (2)$$

For T exponential with rate λ , it is not hard to show that

$$ES_e = \frac{1 - E(e^{-\lambda S})}{\lambda E(e^{-\lambda S})} \quad (3)$$

and hence for stability we need $E(e^{-\lambda S}) > \mu/(\mu + 1)$.

For the $M^X/G/1$ LCFS-p-repeat queue, we can identify the following embedded random walk. The number in the system at arrival and departure epochs during a busy period is equivalent to a random walk on the nonnegative integers with absorbing state 0. The random walk starts at the random point X_0 , decreases by 1 if $T > S$ (a departure), and increases if $T < S$ (an arrival). When it increases, it increases by X , where X is independent of S and T . Thus, we have the following characterization of M , where $I = 1$ if $T < S$ and 0 otherwise, and other definitions are as in previous sections.

Theorem 3.5 *The maximum queue length $M(k)$ for the $M^X/G/1$ queue under the LCFS-p-repeat discipline satisfies*

$$M(k) =_d IM(k + X) + (1 - I) \max\{k, M(k - 1)\},$$

where $M(0) = 0$, and I , X , and $M(k-1)$ are mutually independent, and $M(k+X)$ is independent of I and $M(k-1)$.

Let I' be 1 if $T > S'$, and 0 otherwise. If $P(T > S') \geq P(T > S)$, then $I' \geq_{st} I$. From Theorem 3.5 and a coupling argument it then follows that $M' \leq_{st} M$. Therefore, we have the following.

Theorem 3.6 *For $M^X/G/1$ queues operating under the LCFS-p-repeat discipline, if $E(e^{-\lambda S'}) \geq E(e^{-\lambda S})$, then $M' \leq_{st} M$.*

Note that for the LCFS-p-repeat discipline, we only need for the Laplace transform of the service time evaluated at (the arrival rate) λ to be ordered for two service-time distributions, rather than a complete Laplace-transform ordering. Thus, all possible distributions of service times can be completely ordered, and hence we have a complete stochastic ordering of the corresponding maximum queue lengths. Of course, it is also true that $S' \geq_{cv} S$ implies $S' \leq_{LT} S$, which in its turn implies $E(e^{-\lambda S'}) \geq E(e^{-\lambda S})$.

3.3 LCFS preemptive repeat without resampling

For our final model, we suppose again that when services are preempted they must be restarted from scratch, but now the service time is only drawn from the service-time distribution once. We call this the LCFS-p-repeat-without-resampling discipline. Note that the LCFS-p-repeat and LCFS-p-repeat-without-resampling disciplines are the same for deterministic service times. For stability, we need again $\lambda \mu E S_e(S) < 1$ and $\lambda \mu E S_e(S') < 1$, where $S_e(S)$ is the effective service time. Given $S = s$, the service time is deterministic and the effective service time S_e is the same as in equation (2), that is

$$E[S_e(S)|S = s] = \frac{E(s \wedge T)}{P(s \leq T)} = \frac{1 - e^{-\lambda s}}{\lambda e^{-\lambda s}} = \frac{1}{\lambda}[e^{\lambda s} - 1].$$

Hence,

$$E S_e = E S_s(S) = E(E[S_e(S)|S]) = \frac{1}{\lambda}[E e^{\lambda S} - 1].$$

So for stability we need $E(e^{\lambda S}) < (\mu + 1)/\mu$. If, for example, S is exponentially distributed with mean ν , then for stability we need $\nu > \lambda(\mu + 1)$. Note that this value is larger than for the repeat-with-resampling discipline. Intuitively, a large value of the service time has a large probability of being interrupted and having to start over, and each time it restarts it will again have a large service time.

With X_0 , S_0 , and M defined as in the last subsection, and with T_1 defined to be the first interarrival time after the busy period starts, we now let $M(k, s) = [M|X_0 = k, S_0 = s]$ and $M(k) = M(k, S) = [M|X_0 = k]$. Let $I(s) = 1$ if $T_1 < s$ and 0 otherwise. We have the following.

Theorem 3.7 *The maximum queue length $M(k, s)$ for the $M^X/G/1$ queue under the LCFS- p -repeat-without-resampling discipline satisfies*

$$M(k, s) =_d I(s) \max\{M + k; M(k, s)\} + (1 - I(s)) \max\{k, M(k - 1)\},$$

where $M(0) = 0$, and where $I(s)$, M , $M(k, s)$, and $M(k - 1)$ are independent.

Proof. Given $X_0 = k$ and $S_0 = s$, if an arrival occurs before the first service completion a new i.i.d. (sub-)busy period starts, except that there are k additional customers in the queue. When that sub-busy period ends, the original busy period starts again, independently of T_1 and of M for the ending sub-busy period, with $X_0 = k$ and $S_0 = s$. If the first service completes before an arrival, then we may consider the remainder of the busy period as a new, independent busy period with $k - 1$ initial customers. \square

Let $P(k, b, s) = P(M(k, s) \leq b)$, $P(k, b) = P(M(k) \leq b) = EP(k, b, S_0)$, and $P(b) = P(M < b) = EP(X_0, b, S_0)$. We have the following corollary to Theorem 3.7.

Corollary 3.8 *For $b \geq k \geq 1$ and for all s ,*

$$P(k, b, s) = \frac{P(T > s)P(k - 1, b)}{1 - P(T < s)P(b - k)} = \frac{e^{-\lambda s}P(k - 1, b)}{1 - (1 - e^{-\lambda s})P(b - k)}.$$

Using this corollary, we can show the following.

Theorem 3.9 *For $M^X/G/1$ queues operating under the LCFS- p -repeat-without-resampling discipline, if $S' \leq_{icv} S$, then $M' \leq_{st} M$. Hence, if $S' \geq_{cx} S$, then $M' \leq_{st} M$.*

Proof. From our corollary above, for $b \geq k \geq 1$, and $s \geq 0$,

$$P(k, b, s) = \frac{P(k - 1, b)}{e^{\lambda s}(1 - P(b - k)) + P(b - k)}.$$

It is easy to show that $f(s) := a/(ce^{\lambda s} + d)$ is a decreasing convex function of s for all $a, c, d \geq 0$, $c + d > 0$ (so $-f(s)$ is increasing and concave). Hence, if $S' \leq_{icv} S$ then $-Ef(S') \leq -Ef(S)$ and $Ef(S') \geq Ef(S)$. Also note that $P(k, b, s)$ is increasing in $P(k - 1, b)$ and $P(b - k)$ for fixed s . The result now follows using an induction argument similar to the one in the proof of Theorem 3.4. \square

4 Numerical Illustrations

For the LCFS preempt-resume discipline, we calculated $P(M \leq n)$ using Corollary 3.3. The figures below show the results for several convexly ordered families of distributions, illustrating Theorem 3.4.

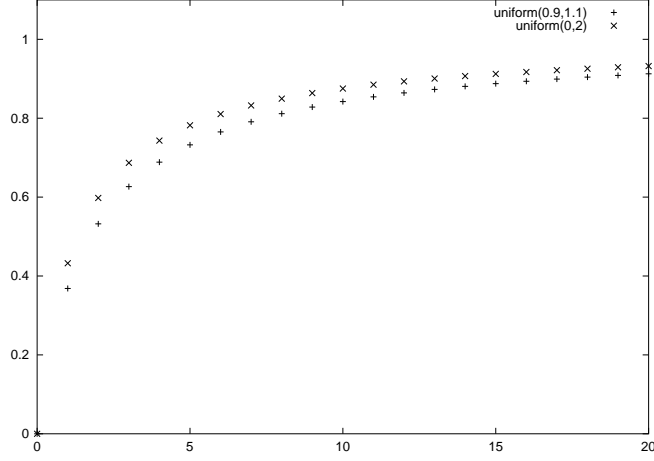


Figure 1: $P(M \leq n), n = 1, 2, \dots, 20$, for uniform service-time distributions with $ES = 1$; $\lambda = 0.9$.

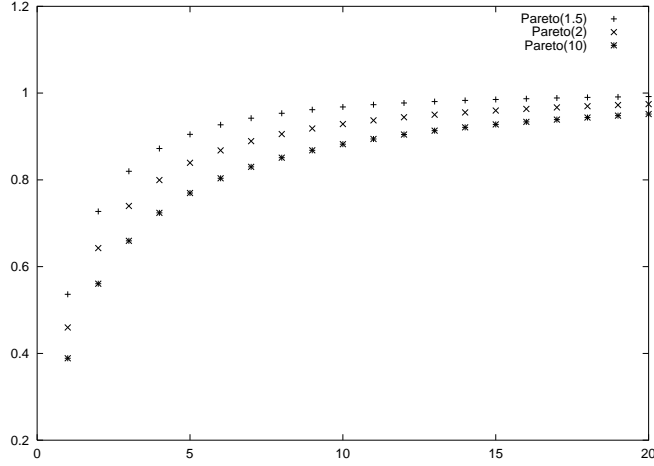


Figure 2: $P(M \leq n), n = 1, 2, \dots, 20$, for Pareto(α) service-time distributions with distribution function $F_\alpha(x) = 1 - ((\alpha - 1)/(\alpha x))^\alpha, x \geq (\alpha - 1)/\alpha$, so $ES = 1$; $\lambda = 0.95$.

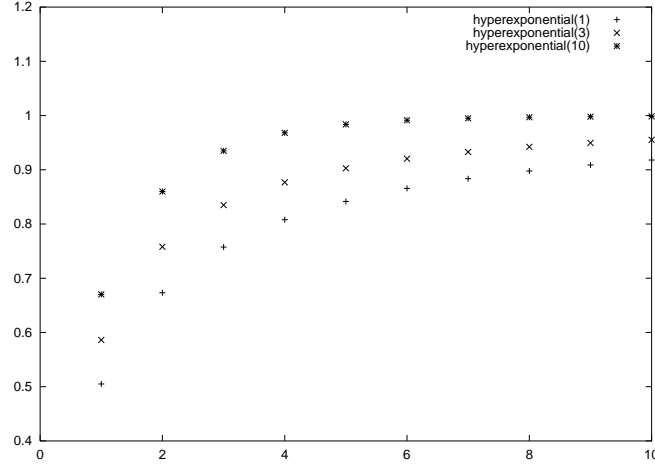


Figure 3: $P(M \leq n), n = 1, 2, \dots, 10$, for hyperexponential(k) service-time distributions with $P(S = X_1) = 1 - 2^{-k} = 1 - P(S = X_2)$, where X_1 and X_2 are exponentially distributed with mean $1/(2(1 - 2^{-k}))$ and $2^k/2$ respectively, so $ES = 1$; $\lambda = 0.95$.

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